

HARMONIC OSCILLATOR IN CHARACTERISTIC p

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ABSTRACT. We construct an irreducible representation of the canonical commutation relations by operators on a certain Banach space over a local field of characteristic p . The Carlitz polynomials forming the basis of the space are shown to be the counterparts of the Hermite functions for this situation. The analogues of coherent states are related to the Carlitz exponential.

1. INTRODUCTION

It was shown in [Ko] that the canonical and deformed commutation relations admit representations by bounded operators in Banach spaces over the field of p -adic numbers, and that various objects related to those representations (analogues of the Hermite functions, coherent states etc.) coincide with well-known special functions of p -adic analysis. The constructions lead to the introduction of some operators (analogues of the number operator) which possess eigenbases orthonormal in the non-Archimidean sense. Note that no general concept of a “Laplacian” is known in p -adic analysis, and p -adic spectral theory (see [B, Vi]) provides no tool to establish the “Hermitian” property of an operator, other than to construct its eigenbasis. Thus the simple difference operators introduced in [Ko] and related to the additive and multiplicative structures of the p -adics, may be viewed as Laplacians on model p -adic domains.

In this Letter we pursue the same line for another model example, the case of a local field of characteristic p , that is of the field K of formal Laurent series with coefficients from the Galois field \mathbf{F}_q . Here p is a prime number, $q = p^\gamma$, $\gamma \in \mathbf{Z}_+$. The construction is very simple again, though quite different from the ones in [Ko], and involves features typical for analysis over K – the basic space is generated by \mathbf{F}_q -linear polynomials, the operators themselves are only \mathbf{F}_q -linear. This time the well-known Carlitz polynomials (see [C1, C2, G1]) appear as analogues of the Hermite functions, and the “coherent states” (eigenfunctions of the annihilation operator) are expressed via the Carlitz exponential, the basic object in the arithmetic of function fields [AT, G2, G3].

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2. CARLITZ BASIS

Let us denote by $|\cdot|$ the non-Archimedean absolute value in K ; if $z \in K$,

$$z = \sum_{i=n}^{\infty} a_i x^i, \quad n \in \mathbf{Z}, \quad a_i \in \mathbf{F}_q,$$

then $|z| = q^{-n}$. Let $O = \{z \in K : |z| \leq 1\}$ be the ring of integers in K ; the ring $\mathbf{F}_q[x]$ of polynomials (in the indeterminate x) with coefficients from \mathbf{F}_q is dense in O . Let K_{ac} be the algebraic closure of K , with the absolute value induced from K . We shall denote by \overline{K} the completion of K_{ac} . It is known [Kü, R] that \overline{K} is an algebraically closed field.

A function $\varphi : O \rightarrow \overline{K}$ is called \mathbf{F}_q -linear if $\varphi(t_1 + t_2) = \varphi(t_1) + \varphi(t_2)$ and $\varphi(\beta t) = \beta \varphi(t)$ for any $t, t_1, t_2 \in O$, $\beta \in \mathbf{F}_q$. Let X be a Banach space (over \overline{K}) of all continuous \mathbf{F}_q -linear \overline{K} -valued functions, with the supremum norm.

The Carlitz polynomials $f_i(t)$, $i = 0, 1, 2, \dots$, are defined as follows. Let $e_0(t) = t$,

$$e_i(t) = \prod_{\substack{m \in \mathbf{F}_q[x] \\ \deg m < i}} (t - m), \quad i \geq 1. \quad (1)$$

It is known [C1, G1] that

$$e_i(t) = \sum_{j=0}^i (-1)^{i-j} \begin{bmatrix} i \\ j \end{bmatrix} t^{q^j} \quad (2)$$

where

$$\begin{bmatrix} i \\ j \end{bmatrix} = \frac{D_i}{D_j L_{i-j}^{q^j}},$$

the elements $D_i, L_i \in K$ are defined as

$$D_i = [i][i-1]^q \dots [1]^{q^{i-1}}; \quad L_i = [i][i-1] \dots [1] \quad (i \geq 1); \quad D_0 = L_0 = 1,$$

and $[i] = x^{q^i} - x \in O$. Finally,

$$f_i(t) = D_i^{-1} e_i(t), \quad i = 0, 1, 2, \dots \quad (3)$$

Since $\text{char } K = p$, we know that $(t_1 + t_2)^{q^j} = T_1^{q^j} + t_2^{q^j}$ for any $t_1, t_2 \in K$; if $\beta \in \mathbf{F}_q$ then $\beta^{q^j} = \beta$. Hence, $f_i \in X$, $i = 0, 1, \dots$.

It was shown by Wagner [W] (see also [G1]) that $\{f_i\}$ is a basis in X , that is any function $\varphi \in X$ admits a unique representation as a uniformly convergent series

$$\varphi = \sum_{i=0}^{\infty} c_i f_i, \quad c_i \in \overline{K}, \quad c_i \rightarrow 0,$$

with $|c_i| \leq 1$ for all i , if $\|\varphi\| \leq 1$.

Proposition 1. *The basis $\{f_i\}$ is orthonormal (in the non-Archimedean sense), that is*

$$\|\varphi\| = \sup_{i \geq 0} |c_i|, \quad \text{for any } \varphi \in X.$$

The proof will follow immediately from the next, more general proposition, which is of some independent interest.

According to [C2, W, G1], the basis $\{f_i\}$ can be complemented up to a basis of the space $C(O, \overline{K})$ of all continuous \overline{K} -valued functions on O , as follows. Let us write any natural number j as

$$j = \sum_{i=0}^{\nu} \alpha_i q^i, \quad 0 \leq \alpha_i < q, \quad (4)$$

and set

$$h_j(t) = \frac{G_j(t)}{\Gamma_j}, \quad G_j(t) = \prod_{i=0}^{\nu} (e_i(t))^{\alpha_i}, \quad \Gamma_j = \prod_{i=0}^{\nu} D_i^{\alpha_i}, \quad (5)$$

so that $h_{q^i} = f_i$, h_j is a polynomial of degree j .

Proposition 2. *The basis $\{h_j\}$ is orthonormal.*

Proof. Let us start from another basis $\{Q_j\}$ in $C(O, \overline{K})$ whose orthonormality follows from the results of [W,A]. Writing any natural number j in the form (4), set

$$Q_0(t) \equiv 1, \quad Q_j(t) = \frac{P_j(t)}{P_j(m_j)}, \quad j \geq 1, \quad (6)$$

where

$$P_j(t) = (t - m_0)(t - m_1) \dots (t - m_{j-1}), \quad m_j = a_{\alpha_0} + a_{\alpha_1}x + \dots + a_{\alpha_{\nu}}x^{\nu},$$

and a_k are the elements of $\mathbf{F}_q = \{a_0, \dots, a_{q-1}\}$, $a_0 = 0$, $a_1 = 1$ (we do not show the dependence of $\nu, \alpha_1, \dots, \alpha_{\nu}$ on j).

Polynomials $\{Q_j\}_{j \leq n}$ form a basis in the space of all polynomials with degrees $\leq n$. In particular, one can write

$$h_n(t) = \sum_{i=0}^n c_{ni} Q_i(t), \quad n = 0, 1, 2, \dots$$

Since $h_n(t) \in \mathbf{F}_q[x]$ for all $t \in \mathbf{F}_q[x]$ (see [C2]), we have $\|h_n\| \leq 1$, whence $|c_{ni}| \leq 1$ for all n, i .

Now, by a general result from [Ve], in order to prove the proposition, it suffices to show that

$$|c_{nn}| = 1, \quad n = 0, 1, 2, \dots$$

It is seen from (1), (3), (5) and (6) that the leading coefficient of $h_n(t)$ equals Γ_n^{-1} while the leading coefficient of $Q_n(t)$ is $(P_n(m_n))^{-1}$ so that $c_{nn} = \Gamma_n^{-1} P_n(m_n)$.

By definition,

$$D_n = x^{1+q+\dots+q^{n-1}} (x^{q^{n-1}} - 1) \left(x^{q^{n-1}-1} - 1 \right)^q \dots (x^{q-1} - 1)^{q^{n-1}},$$

so that $\Gamma_n = x^{l_n} S_n(x)$ where $S_n \in \mathbf{F}_q[x]$, $S_n(0) \neq 0$,

$$l_n = \alpha_1 + (1+q)\alpha_2 + \dots + (1+q+\dots+q^s)\alpha_s,$$

and $\alpha_1, \dots, \alpha_s$ are taken from the q -adic expansion

$$n = \alpha_0 + \alpha_1 q + \dots + \alpha_s q^s, \quad 0 \leq \alpha_j < q. \quad (7)$$

We have $|\Gamma_n| = q^{-l_n}$.

On the other hand,

$$P_n(m_n) = (m_n - m_0)(m_n - m_1) \dots (m_n - m_{n-1}) = x^{\kappa_n} T_n(x),$$

where $T_n \in \mathbf{F}_q[x]$, $T_n(0) \neq 0$, while the number κ_n can be computed as follows. Consider a difference $m_n - m_{n'}$ with $n' < n$, and write

$$n' = \alpha'_0 + \alpha'_1 q + \dots + \alpha'_s q^s, \quad m_{n'} = a_{\alpha'_0} + a_{\alpha'_1} x + \dots + a_{\alpha'_s} x^s.$$

The quantity of the differences $m_n - m_{n'}$ divisible by x equals the quantity of natural numbers of the form $\alpha_0 + \alpha'_1 q + \dots + \alpha'_s q^s$ which are less or equal n (here α_0 is just the same as in (7)), that is the quantity of s -tuples $(\alpha'_1, \dots, \alpha'_s)$ such that

$$\alpha'_1 + \alpha'_2 q + \dots + \alpha'_s q^{s-1} \leq \alpha_1 + \alpha_2 q + \dots + \alpha_s q^{s-1}.$$

Of course, such a quantity equals $\alpha_1 + \alpha_2 q + \dots + \alpha_s q^{s-1}$.

Similarly, the quantity of the differences divisible by x^2 is $\alpha_2 + \alpha_3 q + \dots + \alpha_s q^{s-2}$, so that the number of differences divisible by x and not divisible by x^2 equals to

$$(\alpha_1 + \alpha_2 q + \dots + \alpha_s q^{s-1}) - (\alpha_2 + \alpha_3 q + \dots + \alpha_s q^{s-2}).$$

Continuing this reasoning we find that

$$\begin{aligned} \kappa_n &= (\alpha_1 + \alpha_2 q + \dots + \alpha_s q^{s-1}) - (\alpha_2 + \alpha_3 q + \dots + \alpha_s q^{s-2}) \\ &\quad + 2(\alpha_2 + \alpha_3 q + \dots + \alpha_s q^{s-2}) - (\alpha_3 + \alpha_4 q + \dots + \alpha_s q^{s-3}) + \dots \\ &\quad + (s-1)((\alpha_{s-1} + \alpha_s q) - \alpha_s) + s\alpha_s \\ &= (\alpha_1 + \dots + \alpha_s q^{s-1}) + (\alpha_2 + \dots + \alpha_s q^{s-2}) + \dots + (\alpha_{s-1} + \alpha_s q) \\ &\quad + \alpha_s = l_n \end{aligned}$$

which means that $|c_{nn}| = 1$. \square

3. CARLITZ EXPONENTIAL

The Carlitz exponential is defined by the power series

$$e_C(z) = \sum_{j=0}^{\infty} \frac{z^{q^j}}{D_j}. \quad (8)$$

Since

$$|D_j| = q^{-1}(q^{-1})^q \dots (q^{-1})^{q^{j-1}} = q^{-\frac{q^j-1}{q-1}} = (q^{-1/(q-1)})^{q^j-1},$$

the series (8) is convergent if $z \in \overline{K}$, $|z| < q^{-1/(q-1)}$.

We shall need some relations involving e_C , which coincide formally with the ones well known in number theory (see e.g. [G1] or [AT]), though in the literature the series (8) is usually considered not over \overline{K} , but rather over the ∞ -adic completion of the field of rational functions (with the field of constants \mathbf{F}_q).

Consider a function

$$\rho(\zeta) = \sum_{n=0}^{\infty} (-1)^n \frac{\zeta^{q^n}}{L_n}. \quad (9)$$

It is known [AT] that the functions e_C and ρ are inverse to each other as formal power series. Since $|L_n| = q^{-n}$, the series (9) is convergent if $|\zeta| < 1$, and

$$|\rho(\zeta)| \leq \sup_{n \geq 0} q^n |\zeta|^{q^n}.$$

The function $\psi_\zeta(s) = s|\zeta|^s$ is decreasing for $s > -(\log |\zeta|)^{-1}$; if $|\zeta| < q^{-1/(q-1)}$ then ψ_ζ decreases for $s > (q-1)(\log q)^{-1}$. In particular, $\psi_\zeta(q^n) \leq \psi_\zeta(q)$, $n \geq 1$. Hence, $|\rho(\zeta)| \leq \max(|\zeta|, q|\zeta|^q)$, and we find that

$$|\rho(\zeta)| < q^{-1/(q-1)} \quad \text{if} \quad |\zeta| < q^{-1/(q-1)},$$

which implies the identity

$$e_C(\rho(\zeta)) = \zeta, \quad |\zeta| < q^{-1/(q-1)}. \quad (10)$$

For any fixed $z \in \overline{K}$ such that $|\zeta| < q^{-1/(q-1)}$, consider the function $w_z \in X$ of the form

$$w_z(t) = e_C(tz), \quad t \in O.$$

Let us find its expansion with respect to the Carlitz basis. Below we shall use the difference operators

$$\begin{aligned} \Delta \varphi(t) &\equiv \Delta^{(1)} \varphi(t) = \varphi(xt) - x\varphi(t); \\ \Delta^{(i)} \varphi(t) &= \Delta^{(i-1)} \varphi(xt) - x^{q^{i-1}} \Delta^{(i-1)} \varphi(t), \quad i \geq 2. \end{aligned}$$

Proposition 3. *The function w_z can be expanded as*

$$w_z(t) = \sum_{n=0}^{\infty} (e_C(z))^{q^n} f_n(t). \quad (11)$$

Proof. Let

$$e_C^{(N)}(z) = \sum_{j=0}^N \frac{z^{q^j}}{D_j}; \quad w_{N,z}(t) = e_C^{(N)}(tz).$$

It is clear that $w_{N,z}(t) \longrightarrow w_z(t)$ uniformly with respect to $t \in O$. On the other hand (see [G1]),

$$w_{N,z}(t) = \sum_{n=0}^N b_n^{(N)}(z) f_n(t),$$

where $b_n^{(N)}(z) = \Delta^{(n)} e_C^{(N)}(z)$, $n \leq N$, $\Delta^{(0)} = I$ (the identity operator).

Using (8), the relation $D_i = [i]D_{i-1}^q$, and employing repeatedly the binomial relation for a field of characteristic p , we find by induction that

$$b_n^{(N)}(z) = \left(e_C^{(N-n)}(z) \right)^{q^n}, \quad n \geq 0.$$

It follows from orthonormality of the Carlitz basis that for $N \rightarrow \infty$ the coefficient $b_n^{(N)}(z)$ converges to the n -th coefficient of the expansion of the function w_z . Thus we come to (11). \square

4. CREATION AND ANNIHILATION OPERATORS

The operator $A : X \rightarrow X$ is called \mathbf{F}_q -linear if $A(u+v) = Au + Av$ and $A(\beta u) = \beta Au$ for any $u, v \in X$, $\beta \in \mathbf{F}_q$. The simplest example is the operator $R_q u = u^q$. Its inverse $\sqrt[q]{}$ defined on the set $\{u^q : u \in X\}$ possesses similar properties (recall that $q = p^\gamma$, so that the q -th root, if it exists, is unique in a field of characteristic p).

Now we can introduce our oscillator-like model. The formula (13) below can create an impression that the notation $[i]$ was invented deliberately in order to emphasize the analogy with the conventional quantum mechanics. In reality this notation was proposed by Carlitz in 1935!

Let

$$a^+ = R_q - I, \quad a^- = \sqrt[q]{} \circ \Delta.$$

Theorem. (i) a^+ and a^- are continuous \mathbf{F}_q -linear operators on X ,

$$a^- a^+ - a^+ a^- = [1]^{1/q} I. \tag{12}$$

(ii) The operator $a^+ a^-$ possesses the orthonormal eigenbasis $\{f_i\}$,

$$(a^+ a^-) f_i = [i] f_i, \quad i = 0, 1, 2, \dots; \tag{13}$$

a^+ and a^- act upon the basis as follows:

$$a^+ f_{i-1} = [i] f_i, \quad a^- f_i = f_{i-1}, \quad i \geq 1; \quad a^- f_0 = 0. \tag{14}$$

(iii) The equation

$$a^-u = \lambda u \quad (15)$$

has solutions (“coherent states”) for any $\lambda \in \overline{K}$; if $\lambda \neq 0$, each solution can be written as

$$u(t) = \lambda^{-q/(q-1)} \sum_{n=0}^{\infty} c^{q^n} f_n(t), \quad c \in \overline{K}, \quad |c| < 1, \quad (16)$$

for some value of the $(q-1)$ -th root, and conversely, $a^-u = \lambda u$ for the function (16). If in (16) $|c| < q^{-1/(q-1)}$ then

$$u(t) = \lambda^{-q/(q-1)} e_C(tz), \quad z = \rho(c). \quad (17)$$

In particular, if $q \neq 2$ then every function (16) with $c \in K$ takes the form (17).

Proof. (i) The operator Δ is linear and transforms any function $\varphi \in X$ into the q -th power of some function from X . Indeed,

$$\Delta e_i = \frac{D_i}{D_{i-1}^q} e_{i-1}^q, \quad i \geq 1; \quad \Delta e_0 = 0$$

(see [G1]) whence

$$\Delta f_i = \begin{cases} f_{i-1}^q, & i \geq 1; \\ 0, & i = 0. \end{cases} \quad (18)$$

If

$$\varphi(t) = \sum_{i=0}^{\infty} c_i f_i(t), \quad c_i \rightarrow 0,$$

then (recall that $\text{char } \overline{K} = p$)

$$\Delta \varphi(t) = \sum_{i=1}^{\infty} c_i f_{i-1}^q(t) = \left(\sum_{i=1}^{\infty} c_i^{1/q} f_{i-1}(t) \right)^q.$$

Thus a^- is correctly defined, \mathbf{F}_q -linear, and

$$a^- \varphi(t) = \sum_{i=1}^{\infty} c_i^{1/q} f_{i-1}(t), \quad (19)$$

so that $\|a^- \varphi\| = \sup_{i \geq 1} |c_i^{1/q}| \leq \|\varphi\|$ which implies continuity of a^- . Similar properties of a^+ are obvious.

Simple calculation yields the formulas

$$\begin{aligned} (a^+ a^- \varphi)(t) &= \Delta \varphi(t) - (\Delta \varphi(t))^{1/q}; \\ (a^- a^+ \varphi)(t) &= (\Delta \varphi^q(t))^{1/q} - (\Delta \varphi(t))^{1/q}. \end{aligned}$$

Subtracting we get

$$(a^- a^+ - a^+ a^-) \varphi(t) = (x - x^{1/q}) \varphi(t) = (x^q - x)^{1/q} \varphi(t),$$

and we come to (12).

(ii) The formula (13) is a consequence of (14); the latter follows from (18) and the identities

$$e_i = e_{i-1}^q - D_{i-1}^{q-1} e_{i-1}, \quad D_i = [i] D_{i-1}^q$$

(see [G1]).

(iii) Let $a^- u = \lambda u$,

$$u(t) = \sum_{n=0}^{\infty} c_n f_n(t), \quad c_n \rightarrow 0.$$

It follows from (19) and the uniqueness of the expansion that

$$c_{n+1}^{1/q} = \lambda c_n, \quad n = 0, 1, \dots,$$

whence

$$c_n = \lambda^{q^n + q^{n-1} + \dots + q} c_0^{q^n} = \mu^{-1} (c_0 \mu)^{q^n}, \quad n = 1, 2, \dots,$$

where $\mu = \lambda^{q/(q-1)}$. Since $c_n \rightarrow 0$, we have $|c_0 \mu| < 1$, and we obtain the representation (16) with $c = c_0 \mu$.

The converse statement follows easily from the identity (18).

The representation (17) is a direct consequence of (16) and the results of Sect. 3. If $q \neq 2$, $c \in K$, $|c| < 1$, then $|c| \leq q^{-1} < q^{-1/(q-1)}$, so that in this case (16) is equivalent to (17). \square

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